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Note

A Method for Generation of Orthogonal and Nearly Orthogonal Boundary-Fitted Coordinate Systems

1. Introduction

Conformal mappings which are analytic functions of a complex variable have found extensive application to problems in physics. However, these mappings are restricted to two dimensions and have other limitations which sometimes seriously diminish their usefulness.

In recent years more general transformations have come into use under the names of "boundary-fitted" or "surface-oriented" coordinate systems [1, 2]. These coordinate systems are computed as approximate numerical solutions to elliptic generating equations. Usually the coordinate distribution on the boundaries of the physical region can be specified arbitrarily. The standard conformal system is obtained only when the Laplace equation is used with a particular boundary coordinate distribution. Thus boundary-fitted coordinates encompass a much larger set of systems (including three-dimensional ones) than the subset of conformal mappings. However, these boundary-fitted systems are not, in general, orthogonal. This nonorthogonality can have a deleterious effect on accuracy, stability and computational complexity.

Some effort has been devoted to finding an intermediate set of coordinate systems which retain most of the flexibility of the general boundary-fitted systems but yet are orthogonal. Potter and Tuttle [3] and Ghia *et al.* [4] presented a method for orthogonalizing a nonorthogonal boundary-fitted system. Other methods for generating orthogonal systems were presented in a note by Mobley and Stewart [5] and in work reviewed by Eisemann [6]. However, in all cases the specification of coordinate distribution was somehow restricted on at least one boundary.

The present note discusses a new method for creating orthogonal coordinate systems without this restriction on boundary coordinate distribution. It is shown, through the presentation of examples of successful application to several geometries, that quite simple generating equations can be used for this purpose. A discussion of other, less successful, methods is given in a recent report by the authors [7].

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2. The Construction of Orthogonal Systems

Many investigators have used the system of Poisson equations

$$\begin{aligned} \xi_{xx} + \xi_{yy} &= P, \\ \eta_{xx} + \eta_{yy} &= Q, \end{aligned} \tag{1}$$

to generate numerical transformations from Cartesian coordinates (x, y) to the transformed coordinates (ξ, η) for a variety of geometries. The forcing functions P and Q in Eq. (1) provide a means for influencing or controlling the coordinate system obtained.

For ease in the numerical solution of Eq. (1) all the calculations are done in the transformed plane on a uniform square mesh. Interchanging the dependent and independent variables gives

$$\begin{aligned} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) &= 0, \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) &= 0, \end{aligned} \tag{2}$$

where

$$\alpha = x_\eta^2 + y_\eta^2, \qquad \beta = x_\xi x_\eta + y_\xi y_\eta,$$
$$\gamma = x_\xi^2 + y_\xi^2, \qquad J = x_\xi y_\eta - x_\eta y_\xi.$$

The condition for orthogonality, $\xi =$ constant lines perpendicular to $\eta =$ constant lines, is $\beta = 0$, since

$$\beta = 0 \Rightarrow x_\xi / y_\xi = -y_\eta / x_\eta,$$

which is equivalent to

$$1/y_x \,|_{\eta = \text{constant}} = -y_x \,|_{\xi = \text{constant}}.$$

That is, the slopes of the two sets of coordinate lines are negative reciprocals of each other.

One method for attacking the problem at hand is to attempt to compute forcing functions P and Q which assure an orthogonal mesh for a given geometry and boundary coordinate distribution. The authors tried several methods for doing this by using Eq. (2) directly. These attempts, which met with limited success, are recorded elsewhere [7]. A different approach to this orthogonal grid problem is to bypass the Poisson equations altogether and to choose a generating system based entirely on β. Once the grid points have been suitably positioned in the physical region, the forcing functions for the Poisson system that would generate this configuration could be found from Eq. (2).

Even though $\beta = 0$ is the condition for orthogonality, this equation alone is not sufficient for obtaining the required transformation. Two equations are needed, since we must find both the x and y coordinates of the transformed points. As a new generating system, consider

$$\beta_\xi = \beta_\eta = 0. \tag{3}$$

The solution $\beta =$ constant to Eq. (3) exists only if it is consistent with the boundary data. Only at the corners of the computational region is β specified in advance. Thus we can hope for a solution if β is the same at all corners and for an orthogonal solution only when $\beta = 0$ at the corners.

Expanding Eq. (3) gives

$$x_\xi x_{\xi\eta} + x_{\xi\xi} x_\eta + y_\xi y_{\xi\eta} + y_{\xi\xi} y_\eta = 0, \tag{4a}$$

$$x_\xi x_{\eta\eta} + x_{\xi\eta} x_\eta + y_\xi y_{\eta\eta} + y_{\xi\eta} y_\eta = 0. \tag{4b}$$

To compute the transformation, Eqs. (4a) and (4b) are combined as follows: The product of Eq. (4a) and x_η is added to the product of Eq. (4b) and x_ξ, yielding

$$\begin{aligned} x_\eta^2 x_{\xi\xi} + x_\xi^2 x_{\eta\eta} + 2x_\xi x_\eta x_{\xi\eta} + x_\eta y_\eta y_{\xi\xi} + x_\xi y_\xi y_{\eta\eta} \\ + (x_\eta y_\xi + x_\xi y_\eta) y_{\xi\eta} = 0; \end{aligned} \tag{5a}$$

the product of Eq. (4a) and y_η is added to the product of Eq. (4b) and y_ξ, yielding

$$\begin{aligned} y_\eta^2 y_{\xi\xi} + y_\xi^2 y_{\eta\eta} + 2y_\xi y_\eta y_{\xi\eta} + x_\eta y_\eta x_{\xi\xi} + x_\xi y_\xi x_{\eta\eta} \\ + (x_\eta y_\xi + x_\xi y_\eta) x_{\xi\eta} = 0. \end{aligned} \tag{5b}$$

The reason for replacing Eqs. (4) with Eqs. (5) is to obtain a non-zero coefficient for x_{ij} and y_{ij} in the finite-difference forms of Eqs. (5a) and (5b). This eliminates the possibility of dividing by zero in the iteration process. Each derivative in Eqs. (5a) and (5b) is replaced by the appropriate central difference formula and the system is solved iteratively using successive overrelaxation.

3. Examples

To illustrate the use of this method, we present coordinate systems generated by Eq. (3) for eight different geometries and boundary distributions. In each case the grid shown was obtained by solving the finite difference form of Eq. (5) using successive overrelaxation (SOR). The initial guess for this iterative procedure in all cases was the nonorthogonal grid generated by Eq. (1) with $P = Q = 0$ (a Laplace system). As the first example, Fig. 1 shows a grid generated for a simple rectangular region with nonuniform spacing in both the vertical and horizontal directions. The second of these examples, shown in Fig. 2, is a simply connected region with one convex boundary. Next we have a similar region with a concave rather than convex curved boundary as seen in Fig. 3. Note that the orthogonal mesh must have rather

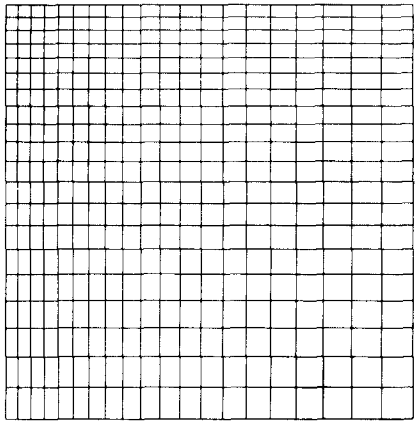

FIG. 1. Orthogonal coordinate system generated for rectangular region by $\beta_\xi = \beta_\eta = 0$.

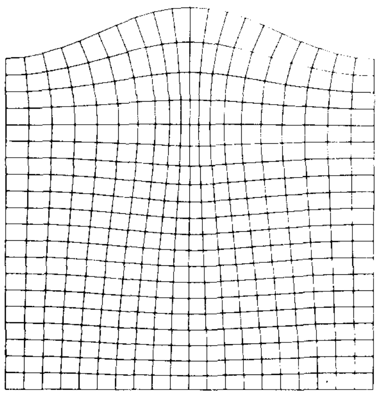

FIG. 2. Coordinate system for region with convex boundary.

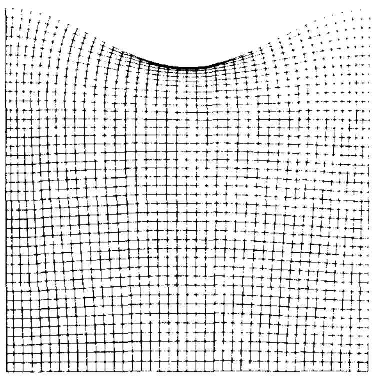

FIG. 3. Coordinate system for region with concave boundary.

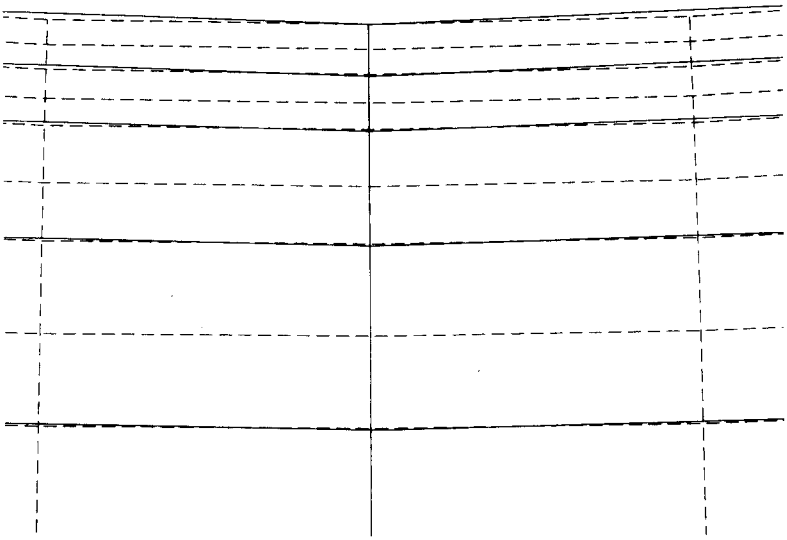

FIG. 4. Comparison of orthogonal grids for region with concave boundary. — 1681 points, --- 6561 points.

fine spacing near the concave upper boundary to accomodate the curvature. During an initial review of this note a question was raised concerning the possibility of a singularity on the upper boundary in the vicinity of the fine mesh spacing. This is a valid question, one that needs to be considered each time the boundary-fitted coordinate technique is used since almost any generating system can produce unacceptable meshes for particular regions. To verify that the fine mesh spacing in Fig. 3 does not indicate a singularity in the transformation, we have refined the mesh. Figure 4 compares two different grids, one coarse with 1681 points and the other fine with 6561 points, generated for the concave region. The fact that corresponding grid lines are in about the same position in both meshes confirms that the coarse

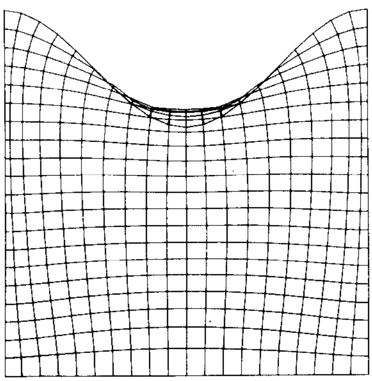

FIG. 5. Grid lines generated for concave region by $\beta_\xi = \beta_\eta = 0$.

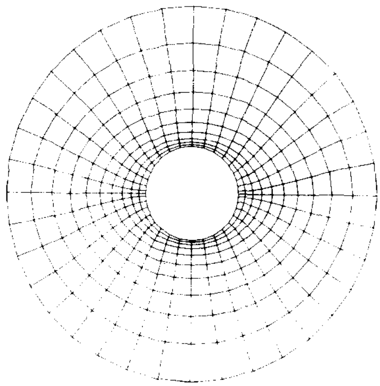

FIG. 6. Orthogonal coordinate system for annular region (horizontal symmetry line).

discretization yields a good approximate solution to the exact problem. A further confirmation comes from consideration of the Jacobian at the midpoint of the upper boundary. The value of $J = 0.00186$ computed on the coarse mesh is seen to agree very well with $J = 0.00047$ on the fine mesh when it is taken into account that these quantities should differ by a factor of four because of the discretization details. There is no indication of a zero Jacobian in the region.

Next we attempted to generate an orthogonal mesh on a region similar to the previous one but with greater curvature of the concave boundary as seen in Fig. 5. Of course an unacceptable mesh such as this one with crossing lines indicates a singular transformation which can often lead to numerical difficulties. But problems like this can also arise from iterative schemes based on the Poisson system if the forcing functions are not chosen carefully. To verify this we computed directly forcing functions P and Q using Eq. (2) with x and y given as in Fig. 5. We then solved Eq. (2) iteratively for x and y using this P and Q, thus regenerating the grid of Fig. 5.

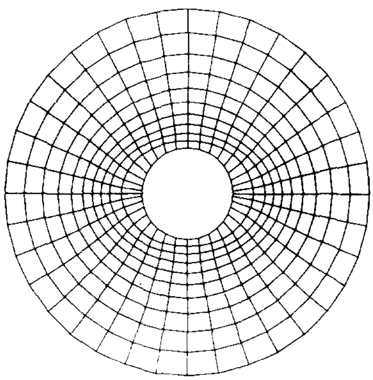

FIG. 7. Orthogonal coordinate system for annular region (vertical symmetry line).

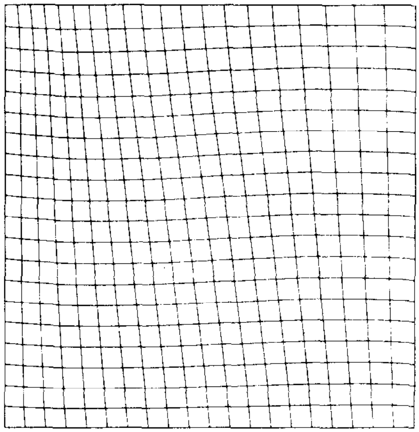

FIG. 8. Coordinate system with unequal spacing on one boundary (441 points).

As the next example, consider a doubly connected region bounded by concentric circles as shown in Figs. 6 and 7. Since this region is symmetric to a line through the center, each grid was generated for half the region and reflected in the line of symmetry. The symmetry line was treated as a boundary with fixed coordinate distribution, thus assuring that $\beta = 0$ at the corners of the computational region. The spacing on the outer boundary, but not on the inner boundary, was uniform. In Fig. 6, the line of symmetry was taken as a horizontal line through the center of the figure while the line of symmetry for Fig. 7 was a vertical line through the center. Interestingly, the two orthogonal grids thus generated (Figs. 6 and 7) are quite dissimilar as a result of different points being held constant after the same initial guess.

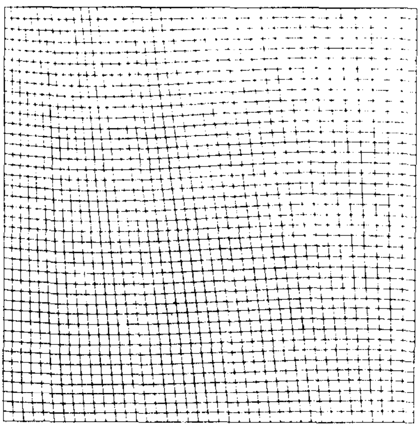

FIG. 9. Coordinate system with unequal spacing on one boundary (1681 points).

For a final test of the method, we consider a square region with the coordinates distributed uniformly on three boundaries, nonuniformly on the fourth. Figure 8 shows a grid of 441 points with $\beta \approx 0.0002$ throughout the field except near the boundaries. In an attempt to improve the orthogonality, the number of grid points was increased to 1681 as seen in Fig. 9. This refinement of the grid did not improve the orthogonality. Thus we believe that no orthogonal mesh exists for this configuration and that the existence of a numerical solution with a small nonzero β results from the weak connection between the interior grid points and the corner points, the only points where $\beta = 0$ is enforced.

4. Conclusion

A new method, based on Eq. (3), which allows arbitrary boundary coordinate distribution, has been used to generate orthogonal and nearly orthogonal systems for several test problems. While the method has limitations and although basic questions remain concerning the existence and uniqueness of orthogonal coordinate systems, this new method adds to the available, useful techniques for constructing these systems. Even a system which is not exactly orthogonal but only nearly orthogonal can be useful in reducing the truncation error which arises in most numerical schemes. Of course, it must be realized that a particular orthogonal coordinate system is not necessarily better than any given nonorthogonal system. In general, it seems beneficial to strive for orthogonality whenever it does not lead to a serious loss of other desirable coordinate system properties.

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References

1. J. F. Thompson, F. C. Thames, C. W. Mastin, and S. P. Shanks, Use of numerically generated body fitted coordinate systems for solution of the Navier–Stokes equations, *in* "AIAA Second Computational Fluid Dynamics Conference, Hartford, Connecticut, June 19–20, 1975."
2. U. Ghia, K. N. Ghia, and C. J. Studerus, Use of surface-oriented coordinates in the numerical simulation of flow in a turbine cascade, *in* "Proceedings, Fifth International Conference on Numerical Methods in Fluid Dynamics" (A. I. van de Vooren and P. J. Zandbergen, Eds.), Lecture Notes in Physics, No. 59, p. 197, Springer-Verlag, Berlin/Heidelberg/New York, 1976.
3. D. E. Potter and G. H. Tuttle, *J. Comp. Phys.* **13** (1973), 483.
4. U. Ghia, J. K. Hodge, and W. L. Hankey, "An Optimization Study for Generating Surface-Oriented Coordinates for Arbitrary Bodies in High-Re Flow," AFFDL-TR-77-117, Air Force Flight Dynamics Laboratory, Wright–Patterson Air Force Base, 1977.

5. C. D. MOBLEY AND R. J. STEWART, *J. Comp. Phys.* **34** (1980), 124–135.
6. P. R. EISEMAN, "Geometric Methods in Computational Fluid Dynamics," ICASE Report No. 80–11 (1980).
7. H. J. HAUSSLING AND R. M. COLEMAN, "Generation of Orthogonal Boundary-Fitted Coordinate Systems," DTNSRDC Report 81/079 (1981).

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H. J. HAUSSLING AND R. M. COLEMAN

Computation, Mathematics and Logistics Department
David W. Taylor Naval Ship Research and
Development Center Bethesda, Maryland 20084